Quantum Ising Dynamics and Majorana Fermions in the Rabi Lattice Model

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The atomic dipoles in the Rabi lattice model exhibit quantum Ising dynamics which is exact in the limit of strong atom-photon interaction. It governs the para- to ferro-electric phase transition in the ground state. As a result of the strong coupling quantum Ising dynamics, the Rabi lattice model on an open chain also realizes two Majorana fermions localized at the boundary ends. In one-dimension, the DMRG calculations of the quantum phase diagram and atomic polarization clearly support the strong coupling behavior, and also yield the correct end-to-end dipole correlation which is a definite signature of these Majorana fermions.

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Introduction.— In recent years, the lattice models of quantum cavities have received much attention for emulating the Bose-Hubbard model by exhibiting the Mott-insulator to superfluid type quantum phase transition for photons [1–5]. A quantum cavity (also called a cavity-QED system) refers to a strongly interacting matter-radiation system inside a high-Q resonator (cavity). The simplest of it can be modeled as a single two-level atom interacting with a single mode of quantized radiation. It has been variously discussed that the lattice models of such quantum cavities may be realized, for example, by engraving an array of cavities in a photonic band-gap material [3, 6], or as a circuit-QED system [7–9]. It is hoped that such engineered systems will be useful in studying interesting quantum many-body problems, much like what the cold atoms in optical lattices do [10].

In complex physical systems, one is often concerned with the phase transitions in the ground state. For example, the superfluid to insulator transition in the ground state of the Bose-Hubbard (BH) model is such a phase transition driven by the competition between the kinetic energy and the local repulsion. Another very famous case which beautifully typifies quantum phase transitions is the quantum Ising (QI) model [11–13]. It describes an Ising system in an external field applied transverse to the Ising direction. The QI model has not only been used to study a variety of physical problems [11, 14–16], but it has also served as an important point of reference in the general understanding of quantum phase transitions [17].

The one-dimensional (1d) spin-1/2 QI model with only nearest-neighbor interactions is an exactly solvable problem that has been rigorously worked out by Pfeuty using the Jordan-Wigner fermionization of the spin-1/2 operators (a method pioneered by Lieb, Schultz and Mattis for the XY chain) [13, 18]. For the 1d QI model, in the fermionized form, Kitaev made a remarkable observation that the two (boundary) ends of an open chain carry a Majorana fermion each [19]. A fermion which is an antiparticle of itself is called a Majorana fermion [19–21]. While there has always been a great interest in finding the Majorana fermions in nature, the developments in quantum computation and condensed matter have further invigorated their search, as clear from the recent experimental activities [22]. Their proposed role in the intrinsically error-free (topological) quantum computation indeed makes the realization of Majorana fermions a very valuable pursuit [19, 23].

In this Letter, we present an interesting case for the realization of Majorana fermions, through the QI dynamics, in the Rabi lattice model. By Rabi lattice we mean a lattice of the Rabi quantum cavities where each cavity has a two-level atom (or a spin-1/2) interacting via dipolar interaction, $\vec{p}\cdot\vec{E}$, with a single mode of quantized radiation, and the inter-cavity coupling amounts to photon hopping. We show that the Rabi lattice model in the limit of strong atom-photon interaction rigorously tends to the QI model, which of course implies that the Rabi lattice in 1d must have two Majorana fermions. This is in contrast to the commonly studied lattice models formed by the Jaynes-Cummings type cavities, which show superfluid-insulator quantum phase transition for photons, but do not exhibit the QI behavior. Below we systematically discuss the emergence of QI dynamics in the Rabi lattice model, and support it by the density matrix renormalization group (DMRG) calculations in 1d.

Rabi lattice model.— A Rabi quantum cavity is a minimal matter-radiation problem of great usefulness. It is described by the Hamiltonian: $\hat{H}_R = \omega(\hat{a}^\dagger\hat{a}+\frac{1}{2})+\frac{\epsilon}{2}\sigma^z+\gamma\sigma^x(\hat{a}^\dagger+\hat{a})$, called the Rabi model. Here, ω is the photon energy and ϵ is the atomic transition energy. The dipole interaction, $\vec{p}\cdot\vec{E}$, is written as $\gamma\sigma^x(\hat{a}^\dagger+\hat{a})$, where γ denotes the atom-photon coupling, $\hat{a}$ ($\hat{a}^\dagger$) is the photon annihilation (creation) operator, and the Pauli operator, $\sigma^x = |e\rangle\langle g| + |g\rangle\langle e|$, measures the atomic dipole moment, $\vec{p}$ [24]. Moreover, $\sigma^z = |e\rangle\langle e| - |g\rangle\langle g|$. The kets $|g\rangle$ and $|e\rangle$ denote the atomic ground and excited levels, respectively. The simplicity of $\hat{H}_R$ is deceptive as its eigenvalue problem has not been solved exactly, although it has been studied extensively [25–27]. A popular variant of this problem, due to Jaynes and Cummings, is however exactly solvable [24, 28].

In the Jaynes-Cummings (JC) model, the atom-photon

interaction is taken as $\gamma(\sigma^+\hat{a} + \sigma^-\hat{a}^\dagger)$, by dropping $\sigma^+\hat{a}^\dagger + \sigma^-\hat{a}$ from the dipole interaction. This is called the rotating-wave approximation (RWA). It is based on a dynamical argument that, close to resonance ($\omega \simeq \epsilon$), the processes corresponding to $\sigma^+\hat{a}^\dagger + \sigma^-\hat{a}$ are faster compared to $\sigma^+\hat{a}+\sigma^-\hat{a}^\dagger$, and hence can be neglected. Here, $\sigma^+ = |e\rangle\langle g|$ and $\sigma^- = |g\rangle\langle e|$. The RWA, however, drastically alters the symmetry properties which gives rise to qualitatively different physical behavior in the two models. In the JC model, it leads to the conservation of 'polariton' number, $\hat{a}^\dagger\hat{a} + \sigma^+\sigma^-$, and makes it analytically solvable. This number conservation [continuous $U(1)$ gauge-symmetry] is not present in the Rabi model. Instead, it is discretely symmetric under the parity operator, $-\hat{\chi}\sigma^z$, where $\hat{\chi} = (-1)^{\hat{n}} = e^{\pm i\pi\hat{n}}$ and $\hat{n} = \hat{a}^\dagger\hat{a}$. The parity is $+1$ for even polariton number and -1 for the odd. Thus, in a Rabi cavity, the states with even polariton number do not mix with the odd ones.

In the lattice models of such quantum cavities, the photons can coherently hop from one cavity to another, while each atom stays put inside the cavity it belongs to. This hopping is a process different from the incoherent decay of photons in a cavity. The photon does not decay while hopping, instead it disperses over the lattice. The time scale of incoherent losses is assumed to be much longer than the time scales associated with the photon hopping and the atom-photon coupling, γ.

Notably, the studies that find the Bose-Hubbard type quantum phase diagram for photons have primarily focussed on the Jaynes-Cummings lattice models. As noted above, the RWA introduces polariton number conservation, due to which the JC lattice model exhibits polariton condensation, just like the material particles of the Bose type. It would be interesting, therefore, to investigate as to what happens without the RWA. Inside a high finesse cavity with strong atom-photon interaction, the counter-rotating terms would invariably influence the physical properties. After all, they are alway there in the dipole interaction. Hence, we study the lattice model of the Rabi quantum cavities.

The Rabi lattice model can be written as follows.

$$\hat{H} = \sum_l \hat{H}_{R,l} - t\sum_{l,\delta}\left(\hat{a}^\dagger_l\hat{a}_{l+\delta} + \hat{a}^\dagger_{l+\delta}\hat{a}_l\right) \qquad (1)$$

Here, $\hat{H}_{R,l} = \omega(\hat{n}_l + \frac{1}{2}) + \frac{\epsilon}{2}\sigma^z_l + \gamma\sigma^x_l(\hat{a}^\dagger_l + \hat{a}_l)$, is the 'local' Hamiltonian of the l^{th} Rabi cavity in the lattice. The second term in $\hat{H}$ describes the nearest-neighbor photon hopping with (amplitude t), and δ denotes the nearest-neighbors (without counting any nearest-neighbor bond twice). For example, on square lattice, $\delta = a\hat{x}$ and $a\hat{y}$ (and not $\pm a\hat{x}$ and $\pm a\hat{y}$). One can also include further neighbor hopping. But, we only consider the nearest-neighbor case. Moreover, the parameters ω, ϵ, γ and t are taken to be positive real numbers.

Note that, for the Rabi lattice model [$\hat{H}$ of Eq. (1)], the parity of the individual cavities is not conserved anymore. This is because the photon hopping incessantly changes the photon occupancy, $\hat{n}_l$, in each cavity which causes the the local parity to fluctuate between odd and even. However, the global parity defined as, $(-)^L\prod_l\hat{\chi}_l\sigma^z_l$, is still a symmetry of the $\hat{H}$. Here, L is the total number of lattice sites (cavities). Since the Rabi lattice model only has a global parity (discrete) symmetry, a thermodynamic transition, if it occurs, would be characterized by the breaking of it, and would not be accompanied by gapless excitations in the 'ordered' phase. This point has been noted recently for the Rabi lattice model [29]. In fact, we expect the Rabi lattice model to exhibit quantum Ising transition, as hinted at by the fluctuating local parity picture presented above. In the following, we clearly establish for this physical scenario by suitably transforming the $\hat{H}$ to a form in which the quantum Ising dynamics expresses more evidently, and explicitly corroborate it by the DMRG calculations in 1d.

Quantum Ising dynamics.— Consider the atom-photon interaction, $\sigma^x_l(\hat{a}^\dagger_l + \hat{a}_l)$, in a Rabi cavity. Since the atomic dipole operator σ^x_l has eigenvalues ±1, we can absorb it into the electric field, $(\hat{a}^\dagger_l + \hat{a}_l)$, by the unitary transformation under $\hat{U} = \prod_{l=1}^L \hat{U}_l$, where $\hat{U}_l = \hat{P}^+_l + \hat{P}^-_l\hat{\chi}_l$ and $\hat{P}^\pm_l = (1 \pm \sigma^x_l)/2$. Under this $\hat{U}$, we get, $\hat{U}^\dagger\hat{a}_l\hat{U} = \sigma^x_l\hat{a}_l$, $\hat{U}^\dagger\hat{n}_l\hat{U} = \hat{n}_l$, $\hat{U}^\dagger\sigma^z_l\hat{U} = \hat{\chi}_l\sigma^z_l$, and $\hat{U}^\dagger\sigma^x_l\hat{U} = \sigma^x_l$. The operators $\hat{n}_l$ and σ^x_l are obviously invariant under $\hat{U}$. Moreover, the $\hat{a}_l$ and $\hat{a}^\dagger_l$ carry σ^x_l, as wanted. Due to this, the dipole interaction transforms to a static 'displacement' term, $\gamma(\hat{a}^\dagger_l + \hat{a}_l)$. Furthermore, the local parity, $-\hat{\chi}_l\sigma^z_l$, transforms to $-\sigma^z_l$. Thus, in the transformed version of the Rabi cavity, the dipole interaction turns into a displacement field for photons, the σ^z_l is to be understood as parity, and σ^x_l continues to be the atomic dipole.

Now consider the Rabi lattice model, $\hat{H}$. Under $\hat{U}$, it transforms to the following form.

$$\hat{U}^\dagger\hat{H}\hat{U} = \hat{H}_1 = \sum_l\left[\omega\left(\hat{n}_l + \frac{1}{2}\right) + \gamma\left(\hat{a}^\dagger_l + \hat{a}_l\right)\right] + \frac{\epsilon}{2}\sum_l\hat{\chi}_l\sigma^z_l - t\sum_{l,\delta}\sigma^x_l\sigma^x_{l+\delta}\left(\hat{a}^\dagger_l\hat{a}_{l+\delta} + \hat{a}^\dagger_{l+\delta}\hat{a}_l\right) \qquad (2)$$

The role of hopping in causing local parity fluctuations is more evident now due to the explicit $\sigma^x_l\sigma^x_{l+\delta}$ factors. Although the transformed Hamiltonian, $\hat{H}_1$, appears to be more complex, it actually helps in making systematic progress. Look at the atom-photon interaction. It acts as a constant displacement field for the photons, which guarantees that $\langle\hat{a}_l\rangle \neq 0$ in the ground state. Thus, a static 'electric' field, proportional to γ, is ever present. It has interesting consequences for the parity and dipole dynamics. This static field, through hopping, would generate a static 'Ising' interaction between the atomic dipoles, σ^x_l. Moreover, the atomic energy, ϵ (renormalized by the expectation of $\hat{\chi}_l$), would act as a 'transverse' field on the parity, σ^z_l. Thus emerges the quantum Ising dynamics in

the Rabi lattice model. While the photon fluctuations above the static field would also be present, in the strong coupling limit ($\gamma \gg$ the other energy scales), they can be neglected.

To give this discussion a proper theoretical form, we do a displacement operation on $\hat{H}_1$ such that the $\gamma(\hat{a}_l^\dagger + \hat{a}_l)$ is absorbed into $\hat{n}_l$. This is done by the unitary operator, $\hat{D} = \prod_{l=1}^{L} \hat{D}_l$, where $\hat{D}_l = e^{-\frac{\gamma}{\omega}(\hat{a}_l^\dagger - \hat{a}_l)}$. Since $\hat{D}^\dagger \hat{a}_l \hat{D} = \hat{a}_l - \frac{\gamma}{\omega}$ and $\hat{D}^\dagger \hat{\chi}_l \hat{D} = \hat{\chi}_l \, e^{-\frac{2\gamma}{\omega}(\hat{a}_l^\dagger - \hat{a}_l)}$, the $\hat{H}_1$ transforms to the following form under $\hat{D}$.

$$\hat{D}_0^\dagger \hat{H}_1 \hat{D}_0 = \hat{H}_2 = \sum_l \left[\omega \left(\hat{n}_l + \frac{1}{2} \right) - \frac{\gamma^2}{\omega} \right] + \frac{\epsilon}{2} e^{-\frac{2\gamma^2}{\omega^2}} \sum_l \hat{\chi}_l \, e^{-\frac{2\gamma}{\omega}\hat{a}_l^\dagger} e^{\frac{2\gamma}{\omega}\hat{a}_l} \, \sigma_l^z - 2t\frac{\gamma^2}{\omega^2} \sum_{l,\delta} \sigma_l^x \sigma_{l+\delta}^x - t \sum_{l,\delta} \sigma_l^x \sigma_{l+\delta}^x \left\{ \left[\hat{a}_l^\dagger \hat{a}_{l+\delta} - \frac{\gamma}{\omega} \left(\hat{a}_l^\dagger + \hat{a}_{l+\delta} \right) \right] + h.c. \right\} \quad (3)$$

This is the Rabi lattice model written explicitly in terms of the static field, $\frac{\gamma}{\omega}$, and the photon fluctuations. Now, invoke the strong coupling limit, $\frac{\gamma}{\omega} \gg 1$. It amounts to neglecting the photon operator terms in $\hat{H}_2$. The limiting strong coupling problem that results is the quantum Ising model of parities and dipoles written below.

$$\hat{H}_{QI} = \frac{\epsilon}{2} e^{-\frac{2\gamma^2}{\omega^2}} \sum_l \sigma_l^z - 2t\frac{\gamma^2}{\omega^2} \sum_{l,\delta} \sigma_l^x \sigma_{l+\delta}^x \quad (4)$$

Here, the Ising interaction, $J = -2t\frac{\gamma^2}{\omega^2}$, between atomic dipoles is ferro-electric, and the transverse field acting on the local parities, $h = \frac{\epsilon}{2}\exp(-\frac{2\gamma^2}{\omega^2})$, is exponentially suppressed (akin to the Ham, or Frank-Condon, reduction factors in vibronic states [30, 31]).

Ground state properties.— Since the QI model is a well-studied problem, having $\hat{H}_{QI}$ as the limiting Hamiltonian immensely helps in understanding the ground state of the Rabi lattice model. This is more so when γ is the largest energy scale (the domain of applicability of $\hat{H}_{QI}$), but also in general. It predicts two distinct phases, characterized by the atomic polarization, that undergo quantum Ising transition between them.

For $|J| \gtrsim h$, the Rabi lattice would exhibit spontaneous polarization. In this ferro-electric (FE) ground state, the order parameter, $p = \frac{1}{L}\sum_l \langle \sigma_l^x \rangle \neq 0$. For $|J| \lesssim h$, each Rabi cavity on the lattice would (roughly) behave as independent, and $p = 0$ [para-electric (PE) phase]. This can also be restated in terms of photons, as often done in the literature on the cavity lattice models. The photon order parameter, $\psi = \langle \hat{a}_l \rangle$, can be described (after having applied $\hat{U}$ and $\hat{D}$) as $\psi = \langle \sigma_l^x (\hat{a}_l - \frac{\gamma}{\omega}) \rangle \approx -p\frac{\gamma}{\omega}$. Thus, in the FE phase, the photons exhibit 'superfluidity'. The PE phase may likewise be called a Mott insulating phase. However, it not quite the same because a Mott phase is characterized by an integer polariton number per cavity, which is not true for the Rabi lattice. Moreover, the FE phase is also not the usual superfluid described by a complex $U(1)$ order parameter with gapless modes. Instead, it has a real Ising (Z_2) order parameter with gapped excitations. Notably the excitations in both the phases are gapped. They can be viewed as parity flips in the PE phase, and atomic dipole flips in the FE phase. The closing of the gap marks the quantum phase transition.

The factor, $\exp(-\frac{2\gamma^2}{\omega^2})$, in the transverse field, h, too has some important consequences. For $\frac{\gamma}{\omega} \gg 1$, it renders the variations in ϵ unimportant, as h would invariably be small regardless of the value of ϵ. For the same reasons, it is immaterial whether ϵ is near resonance ($\sim \omega$) or far. The smallness of h also allows the Ising interaction, J, to dominate because of which the critical hopping, t_c, for the transition from PE to FE phase would be small.

To explicitly validate the strong coupling behavior, and to see deviations from it, we numerically investigate the Rabi lattice model in 1d using DMRG. The DMRG is a very nice method for studying the ground state properties of 1d quantum problems [32, 33]. Since the $\hat{H}_{QI}$ in 1d is exactly solvable, it also allows us to make proper comparisons with the DMRG results. Notably, through the strong coupling QI problem, the 1d Rabi lattice also presents an exciting possibility for realizing Majorana fermions, which indeed makes it a special case to study.

For $\hat{H}_{QI}$ in 1d, the exact polarization in the FE phase is $p = (1 - \frac{1}{\lambda^2})^{\frac{1}{8}}$ (in the thermodynamic limit) [13], where $\lambda = |J|/h = \frac{4t}{\epsilon}\frac{\gamma^2}{\omega^2}\exp(\frac{2\gamma^2}{\omega^2})$. Using DMRG, we compute p as a function of t in the ground state of the Rabi lattice model (in the $\hat{H}_2$ form). It clearly exhibits the PE to FE phase transition, and agrees nicely with the strong coupling QI form, as shown in Fig. 1. This DMRG data is computed for a chain of length $L = 204$, and by taking four photon states ($n_l = 0, 1, 2, 3$) per Rabi cavity.

Since the exact quantum critical point for the PE to FE phase transition in $\hat{H}_{QI}$ in 1d is $\lambda = 1$ [13], the strong coupling critical hopping is $t_c = \frac{\epsilon}{4}\frac{\omega^2}{\gamma^2}\exp(-\frac{2\gamma^2}{\omega^2})$. The t_c for $\epsilon = \omega$ that has been derived recently in Ref. 34 is just

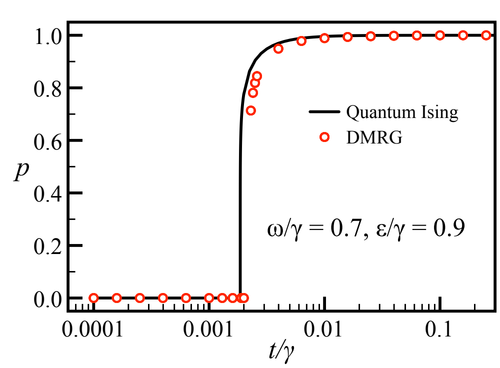

FIG. 1. Polarization, p, vs. hopping, t, in 1d. The black continuous line is the exact strong coupling prediction.

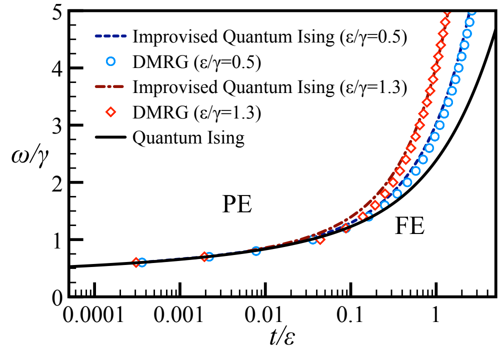

FIG. 2. Quantum phase diagram for the Rabi lattice model in 1d. The PE (FE) denotes the para- (ferro-) electric phase. The continuous line is the strong coupling phase boundary.

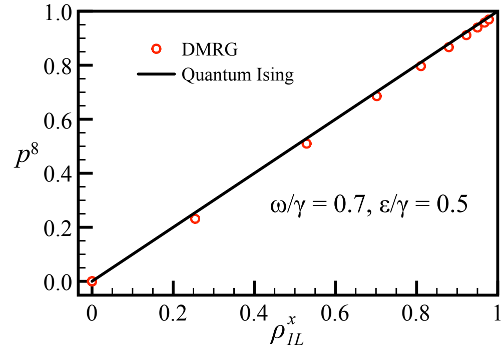

FIG. 3. End-to-end correlation, ρ^x_{1L}, vs. p^8 on an open chain.

a particular case of the present result. Note that, for a given $\frac{\gamma}{\omega}$, the strong coupling t_c scales as ϵ, and it is true in any dimension. This implies that, in the plane of $\frac{t}{\epsilon}$ and $\frac{\omega}{\gamma}$, the strong coupling phase boundary would be a universal curve, independent of ϵ. In Fig. 2, the DMRG calculated phase boundary is compared with the strong coupling prediction. Clearly, for $\frac{\gamma}{\omega} \gtrsim 1$, the numerical data approaches the universal t_c/ϵ derived above.

For $\frac{\gamma}{\omega} \lesssim 1$, however, there are clear deviations from the strong coupling QI behavior. We can approximately account for these deviations within a simple-minded improvisation of the $\hat{H}_{QI}$, which corresponds to replacing ω by $\omega - zt\rho^x_1$ in Eq. (4). It is basically correcting ω for the photon dispersion. Here, z is the nearest-neighbor coordination (which is 2 in 1d), and $\rho^x_1 = \langle \sigma^x_l \sigma^x_{l+\delta} \rangle$ is the nearest-neighbor correlation. This improvisation is effected by a more general displacement operation, $\prod_l e^{-\alpha(\hat{a}^\dagger_l - \hat{a}_l)}$, on $\hat{H}_1$, and approximating $t\alpha \sum_{l,\delta} \sigma^x_l \sigma^x_{l+\delta} (\hat{a}_l + \hat{a}_{l+\delta} + h.c.)$ (arising from the photon hopping after applying the displacement) by $zt\alpha\rho^x_1 \sum_l (\hat{a}^\dagger_l + \hat{a}_l)$. Putting the terms linear in photon operators to zero gives $\alpha = \gamma/(\omega - zt\rho^x_1)$, and ignoring the photon fluctuations gives the improvised $\hat{H}_{QI}$. Thus, we get the improvised QI phase boundaries shown in Fig. 2, whose agreement with the DMRG data is quite reasonable. Approximating $\sigma^x_l \sigma^x_{l+\delta}$ by ρ^x_1 is particularly okay near the transition where ρ^x_1 is strong (supported by DMRG calculations of ρ^x_1). At the transition point, $\rho^x_1 = \frac{2}{\pi}$ for the 1d QI model. The improvised QI model may not work for much weaker couplings.

Majorana fermions.— On an open chain, the strong coupling $\hat{H}_{QI}$ guarantees that the Rabi lattice has two 'unpaired' Majorana fermions in the FE phase. This is because the exact excitations of the QI model in 1d behave as fermions, of which, one corresponds to having two Majorana fermions localized at the two ends in the ordered phase [13, 19]. Another important feature concerns the exact end-to-end correlation, $\rho^x_{1L} = \langle \sigma^x_1 \sigma^x_L \rangle = p^8 + \mathcal{O}(1/L)$, in the ground state of the open chain QI model [13]. The fact that ρ^x_{1L} is p^8 (in the thermodynamic limit) is at odds with the normal expectation, p^2. This striking property of ρ^x_{1L} is a direct consequence of these Majorana fermions [18], and thus a definite signature thereof. Our calculations clearly find this behavior in the Rabi lattice model for large couplings (see Fig. 3). From an experimental perspective, Fig. 3 is quite interesting because it compares two observables only (end-to-end dipole correlation and spontaneous polarization), and doesn't explicitly involve any model parameters.

Summary.— The properties of the Rabi lattice model crucially depend upon the dipole interaction and photon hopping. For the dipolar atom-photon interaction, the parity is the only good quantum number locally in each cavity. The photon hopping, however, causes these local parities to fluctuate. This is aptly described by $\hat{H}_1$, in which the atomic dipoles, through photon hopping, compete with the parities. This identification is facilitated by the unitary transformation, $\hat{U}$, which helps the photons to carry the discrete Z_2 phase of the dipoles. It also transforms the atom-photon interaction to a static displacement field for photons. This leads to a crucial insight that if the atom-photon coupling is very strong, then the photons will behave as a constant 'electric' field with negligible fluctuations. This is effected by rewriting the problem in the displaced photon basis as $\hat{H}_2$, which in the strong coupling limit, $\frac{\gamma}{\omega} \gg 1$, tends to $\hat{H}_{QI}$, the quantum Ising model of dipoles and parities. This drives the PE to FE quantum phase transition, characterized by the spontaneous polarization, p, of atomic dipoles, in the Rabi lattice model. In the FE phase on an open chain, the strong coupling QI dynamics guarantees the existence of two Majorana fermions located at the two ends. This is an important finding in view of the current excitement about Majorana fermions. The end-to-end dipole correlation equals p^8 is a clear signature of the Majorana fermions to look for experimentally. The theoretical analysis of the Rabi lattice model is well supported by the numerical DMRG calculations in 1d.

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